

Simple Derivation of Binet-Minc Formula and Ryser Formula on the Evaluation of Permanents

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1. Introduction

Among several formulas on the evaluation of permanents, there are two beautiful ones. One is due to Ryser[6], and the other is so called Binet-Minc formula (Minc[5]). Ryser applied the principle of inclusion and exclusion to obtain his formula. Crapo[3] derived these formulas using the Möbius functions. Bebiano[2] used differential operator to derive these formulas. Marcus and Sandy[4] gave a proof of Ryser formula based on multilinear algebra.

In this note we give an elementary proofs of these formulas. Evaluating the permanent of a particular matrix, we derive Binet-Minc formula. For Ryser formula, we use an expression for the Stirling number of the second kind. About the notations and terminologies, we mainly follow the introductory part of the book by Aigner[1].

2. Notation

Let N be the set of integers: $N = \{1, 2, \dots, n\}$. A partition of the set N into p blocks N_1, \dots, N_p is called a *set-partition*. Set-partitions are denoted by lowercase Greek letters π, σ, \dots . The totality of set-partitions, L_N , has a natural semi-order induced by *refinement relation*, which is denoted by " \leq " or " \geq ". The set N itself is the unique maximal, I , of L_N , and the unique minimal, O , is the set-partition consisting of n singleton blocks, $\{1\}, \dots, \{n\}$. (c.f. Aigner[1], p. 13)

On the other hand, a non-increasing sequences of $p (\leq n)$ positive integers $j = (j_1, \dots, j_p)$ is called a *number-partition* of the integer n if $j_1 + \dots + j_p = n$. These integers are called *parts* of j , and p is called *length* of j . The set of all number-partitions of n with length p is denoted by $G_n(p)$ and the union of $G_n(1), \dots, G_n(n)$ is denoted by G_n .

Given a set-partition π of the set N into p blocks N_1, \dots, N_p , their cardinalities arranged in non-increasing order form a number-partition of n into p parts.

3. Binet-Minc Formula

Let $A = (a_{ij})$ be an $n \times m$ matrix. For a set-partition $\pi = (N_1, \dots, N_p)$ of N , let

$$S_A(\pi) = \prod_{t=1}^p \sum_{j=1}^m \prod_{i \in N_t} a_{ij} \quad (1)$$

and also let

$$T_A(\pi) = \sum \prod_{t=1}^p \prod_{i \in N_t} a_{iht} \quad (2)$$

where the summation is for all p -permutations $(h_1 \cdots h_p)$ of the set of integers $M = \{1, 2, \cdots, m\}$. When $p > m$, we set $T_A(\pi) = 0$, since there is no p -permutation. When $n \leq m$ and $\pi = (\{1\}, \cdots, \{n\})$, namely, $\pi = O$ in L_N , $T_A(0)$ is nothing but the permanent of the matrix $A = (a_{ij})$.

Given a number-partition $\underline{j} \in G_n(p)$, let $L_N(\underline{j})$ be the totality of set-partitions having \underline{j} as their corresponding number-partition. Let $S_A(\underline{j})$ and $T_A(\underline{j})$ be the sum of $S_A(\pi)$ in (1) and $T_A(\pi)$ in (2), respectively, for all the set-partitions $\pi \in L_N(\underline{j})$. Note the relation $T_A(0) = T_A(\underline{n})$ where $\underline{n} = (1, \cdots, 1)$, and the famous Binet-Minc formula expresses it as a linear form of $S_A(\underline{k})$ for $\underline{k} \in G_n$.

Theorem 1. (Binet-Minc) Let $A = (a_{ij})$ be an $n \times m$ matrix.

Then

$$\text{Per}(A) = \sum_{q=1}^n \sum_{\underline{k} \in G_n(q)} c(\underline{k}) S_A(\underline{k}) \quad (3)$$

where $\underline{k} = (k_1, \cdots, k_q) \in G_n(q)$ and

$$c(\underline{k}) = (-1)^{n-q} \prod_{i=1}^q (k_i - 1)!. \quad (4)$$

Proof. It is clear, from the definitions of $S_A(\pi)$ and $T_A(\pi)$, we have for any $\pi \in L_N$

$$S_A(\pi) = \sum_{\sigma \geq \pi} T_A(\sigma) \quad (5)$$

where the summation is for all $\sigma \in L_N$ with $\sigma \geq \pi$. These relations may be regarded as simultaneous linear equations, whose matrix is triangular. Its diagonal elements are all equal to 1, and hence it is invertible. Note that the inverse matrix depends only on L_N and is not only free from the values of a_{ij} but also $m (\geq n)$. The solution of the linear equations guarantees the existence of numbers $c(\sigma)$ such that

$$\text{Per}(A) = T_A(0) = \sum_{\sigma \in L_N} c(\sigma) S_A(\sigma). \quad (6)$$

In order to evaluate $c(\sigma)$, we proceed as follows.

Take a number-partition $\underline{k} = (k_1, \cdots, k_q) \in G_n(q)$ with $1 \leq q \leq n$. Let

$$\sigma_0 = \{N_1, \cdots, N_q\}$$

where

$$N_i = \{k_1 + \cdots + k_{i-1} + 1, \cdots, k_1 + \cdots + k_i\}. \quad (i = 1, \cdots, q)$$

Let m_1, \cdots, m_q be positive integers with $m_i \geq k_i$ ($i = 1, \cdots, q$) and consider a matrix V given by

$$V = \begin{pmatrix} J(1) & 0 & \cdots & 0 \\ 0 & J(2) & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & J(q) \end{pmatrix}$$

where $J(i)$ is the $k_i \times m_i$ matrix whose elements are all 1. Obviously the left hand side of (6) is

$$\text{Per}(V) = \prod_{i=1}^q m_i (m_i - 1) \cdots (m_i - k_i + 1). \quad (7)$$

On the other hand, the right hand side is

$$c(\sigma_0) m_1 \cdots m_q + \sum_{\sigma < \sigma_0} c(\sigma) M(\sigma) \quad (8)$$

where $M(\sigma)$ is a monomial of higher order than $m_1 \cdots m_q$. As the equality holds for all m_i , with $m_i \geq k_i$ ($i=1, \dots, q$), $c(\sigma_0)$ is equal to the right hand side of (4).

The similar argument can be applied to any set-partition σ having the same number-partition \underline{k} by permuting the rows of V in accordance with the blocks of σ . Thus $c(\sigma)$ is common for all $\sigma \in L_N(\underline{k})$, and as $S_A(\underline{k})$ is the sum of such $S_A(\sigma)$ we have (3). (q. e. d.)

4. Ryser formula

Let $A = (a_{ij})$ be an $n \times m$ matrix. For a non-empty subset, say L , of $M = \{1, \dots, m\}$, we define

$$\Phi_A(L) = \sum \prod_{i=1}^n a_{if(i)} \quad (9)$$

where summation is over all surjective mappings f from the set $N = \{1, \dots, n\}$ to L . If $|L| > n$ or $L = \emptyset$, we define $\Phi_A(L) = 0$. Further let

$$\Psi_A(L) = \prod_{i=1}^n \sum_{j \in L} a_{ij}. \quad (10)$$

For convenience we define $\Psi_A(\emptyset) = 0$.

When $n=m$, $\Phi_A(M)$ becomes the permanent of A .

Theorem 2. (Ryser) Let $A = (a_{ij})$ be an $n \times m$ matrix. Then

$$\text{Per}(A) = \sum_{L \subseteq M} \sum_{\substack{K \subseteq L \\ |K|=n}} (-1)^{n-|K|} \Psi_A(K) \quad (11)$$

$$= \sum_{\substack{K \subseteq M \\ |K| \leq n}} (-1)^{n-|K|} \binom{m-|K|}{n-|K|} \Psi_A(K) \quad (12)$$

Proof. It is clear, from the definitions of $\Phi_A(L)$ and $\Psi_A(L)$, we have for any $L \subseteq M$

$$\Psi_A(L) = \sum_{K \subseteq L} \Phi_A(K). \quad (13)$$

Here the summation is over all subsets of L . These relations may be regarded as simultaneous linear equations, whose matrix is triangular. Its diagonal elements are all equal to 1, and hence it is invertible. Note that the inverse matrix depends only on the set M and is not only free from the values of a_{ij} but also n . The solution of the linear equations guarantees the existence of numbers $d(K, M)$ such that

$$\Phi_A(M) = \sum_{K \subseteq M} d(K, M) \Psi_A(K). \quad (14)$$

It is easily seen that the numbers, $d(K, M)$, depend only on the cardinalities $|K|$ and $|M|$.

In order to evaluate these numbers, take the $n \times m$ matrix E all of whose entries equal to 1. Then we have

$$\Psi_E(K) = |K|^n. \quad (15)$$

It is well known (c.f. [1], p. 70 and p. 97)

$$\begin{aligned} \Phi_E(M) &= \text{the cardinality of the set of all surjective mappings from } N \text{ to } M \\ &= m! S_{n,m} \\ &= \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} k^n \end{aligned} \quad (16)$$

where $S_{n,m}$ is the Stirling number of the second kind.

Substituting (15) and (16) into (14), it follows that

$$\sum_{k=1}^m (-1)^{m-k} \binom{m}{k} k^n = \sum_{k=1}^m \left\{ \sum_{\substack{K \subseteq M \\ |K|=k}} d(K, M) \right\} k^n. \quad (17)$$

Thus we have $d(K, M) = (-1)^{|M|-|K|}$. Finally noting the relation

$$\text{Per}(A) = \sum_{\substack{L \subseteq M \\ |L|=n}} \Phi_A(L),$$

we arrive at the equality (12) in the Theorem. (q.e.d.)

References

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